MATHEMATICS
Paper – II

| Time Allowed : **Three** Hours | Maximum Marks : **200** |

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are **compulsory**. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

*Answers must be written in **ENGLISH** only.*

SECTION A

Q1. (a) Let F be a finite field of characteristic p, where p is a prime. Then show that there is an injective homomorphism from $\mathbb{Z}_p$ (group of integers modulo p) to F. Also show that number of elements in F is p^n, for some positive integer n. *8*

(b) Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{Q}$ denote the set of rational numbers. If $x \in \mathbb{R}$, $x > 0$ and $y \in \mathbb{R}$, then show that there exists a positive integer n such that $nx > y$. Use it to show that if $x < y$, then there exists $p \in \mathbb{Q}$ such that $x < p < y$. *8*

(c) Suppose $f : [a, b] \to \mathbb{R}$ is a continuous function. Then show that f is Riemann integrable on [a, b]. *8*

(d) Prove that the linear programming problem

Maximize
$$z = 3x_1 + 2x_2$$
subject to the constraints:
$$2x_1 - x_2 \leq 2$$
$$3x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0$$

does not admit an optimum basic feasible solution. *8*

(e) Compute the integral

$$\oint_C \frac{1 + 2z + z^2}{(z - 1)^2 (z + 2)} dz$$

where C is $|z| = 3$. *8*

Q2. (a) Find all the Sylow p-subgroups of S_4 and show that none of them is normal. *10*

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(b) Suppose $\{f_n\}$ is a sequence of functions defined on [a, b] and $\lim\limits_{n \to \infty} f_n(x) = f(x)$, and $x \in$ [a, b]. Put $M_n = \sup\limits_{x \in [a, b]} |f_n(x) - f(x)|$.

Then show that

(i) f_n converges to f uniformly on [a, b] if and only if $M_n \to 0$ as $n \to \infty$.   7

(ii) If $|f_n(x)| \le M_n$, ($x \in$ [a, b], n = 1, 2, ...), then $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly on [a, b] if $\sum\limits_{n=1}^{\infty} M_n$ converges.   8

(c) Find a bilinear transformation w = f(z) which maps the upper half plane $\mathrm{Im}(z) \ge 0$ onto the unit disk $|w| \le 1$.   15

Q3. (a) (i) Prove that every bounded and monotonically increasing sequence is convergent and converges to lub (least upper bound) of the sequence.   5

(ii) If $a_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + ... + \dfrac{1}{n}$, $\forall\, n \in \mathbb{N}$, then using Cauchy criterion for convergence of the sequence, show that $\{a_n\}$ is not convergent.   5

(b) (i) Let P be a Sylow p-subgroup of a group G and H is any p-subgroup of G such that HP = PH. Then show that $H \subseteq P$.   7

(ii) Show that every group of order 15 is cyclic.   8

(c) Employ duality to solve the following linear programming problem :   15

Maximize
$$z = 2x_1 + x_2$$
subject to the constraints :
$$x_1 + 2x_2 \le 10$$
$$x_1 + x_2 \le 6$$
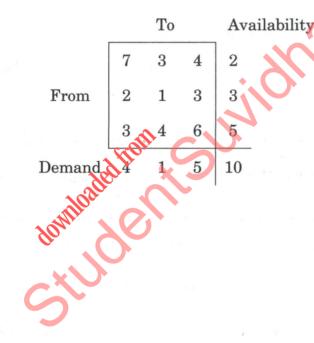$$x_1 - x_2 \le 2$$
$$x_1 - 2x_2 \le 1$$
$$x_1, x_2 \ge 0$$

Q4. (a) (i) Find an upper bound for the absolute value of the integral

$$I = \int_C e^z \, dz,$$ where C is the line segment joining the points $(0, 0)$ and $(1, 3)$. *8*

(ii) Find the length of the curve C defined by

$z(t) = (1 - 2i)t^3, \, -1 \le t \le 1.$ *7*

(b) Prove that R[x] is a principal ideal domain if and only if R is a field. *10*

(c) Find the initial basic feasible solution to the following transportation problem by the North-West corner rule and then optimize it. *15*

	To			Availability
	7	3	4	2
From	2	1	3	3
	3	4	6	5
Demand	4	1	5	10

SECTION B

Q5. (a) Equation of any cone with vertex at the point (a, b, c) is of the form $f\left(\dfrac{x-a}{z-c}, \dfrac{y-b}{z-c}\right) = 0$. Find the partial differential equation of the cone. *8*

(b) Given f(1) = 4, f(2) = 5, f(7) = 5 and f(8) = 4. Find the value of f(6) and also the value of x for which f(x) is maximum or minimum. *8*

(c) (i) If x = 0·101010101E0001010 and y = 0·100010110E0000110, then find x − y. *4*

(ii) Draw the map of the Boolean function F = x′yz + xy′z′ + xyz + xyz′. Also simplify the function. *4*

(d) A rod of length 2a revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α. Prove that the direction of reaction at the hinge makes with the vertical, an angle $\tan^{-1}\left[\dfrac{3}{4}\tan\alpha\right]$. *8*

(e) Verify that the equation yz(y + z)dx + xz(x + z)dy + xy(x + y)dz = 0 is integrable and find its solution. *8*

Q6. (a) Find the system of equations for obtaining the general equation of surfaces orthogonal to the family given by

$$x(x^2 + y^2 + z^2) = Cy^2,$$

where C is a parameter. *10*

(b) Write down an algorithm for Simpson's $\dfrac{1}{3}$ rule. Hence, compute $\displaystyle\int_{0}^{1} x^2(1 - x)\,dx$ correct up to three decimal places with step size h = 0·1 and compare the result with its exact value. *6+9*

(c)　If the velocity of an incompressible fluid at the point (x, y, z) is given by (− Ay, Ax, 0), then prove that the surfaces intersecting the stream lines orthogonally exist and are the planes through z-axis, although the velocity potential does not exist. Discuss the nature of the fluid flow.　*15*

Q7.　(a)　Solve the following system of equations by Gauss-Jordan method :　*15*

$$2x + y - 3z = 11$$
$$4x - 2y + 3z = 8$$
$$-2x + 2y - z = -6$$

(b)　Verify that $w = i\,k\,\log\left(\dfrac{z - ia}{z + ia}\right)$ is the complex potential of a steady fluid flow about a circular cylinder, the plane y = 0 being a rigid boundary.

Further show that the fluid exerts a downward force of magnitude $\left(\dfrac{\pi \rho\, k^2}{2a}\right)$ per unit length on the cylinder, where ρ is the fluid density.　*15*

(c)　Find the solution of the partial differential equation

$$z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y); \quad p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$$

which passes through the x-axis, using Cauchy's method of characteristics.　*10*

Q8.　(a)　A particle of unit mass is projected so that its total energy is h in a field of force of which the potential energy is φ(r) at a distance r from the origin. By employing the principle of energy and least action, show that the path is given by the following differential equation :

$$c^2\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right] = r^4\,[h - \phi(r)],$$

where c is a constant.　*15*

(b) Find the real root of the equation $e^x - 3x = 0$, by Newton–Raphson method, correct up to four decimal places. *10*

(c) Find a complete integral of the partial differential equation

$$(p^2 + q^2)x = pz; \quad p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

using Charpit's method and hence deduce the solution which passes through the curve $x = 0$, $z^2 = 4y$. *15*